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Josephson junction of non-Abelian superconductors and non-Abelian Josephson vortices

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Abstract

A Josephson junction is made of two superconductors sandwiching an insulator, and a Josephson vortex is a magnetic vortex (flux tube) absorbed into the Josephson junction, whose dynamics can be described by the sine-Gordon equation. In a field theory framework, a flexible Josephson junction was proposed, in which the Josephson junction is represented by a domain wall separating two condensations and a Josephson vortex is a sine-Gordon soliton in the domain wall effective theory. In this paper, we propose a Josephson junction of non-Abelian color superconductors and show that a non-Abelian vortex (color magnetic flux tube) absorbed into it is a non-Abelian Josephson vortex represented as a non-Abelian sine-Gordon soliton in the domain wall effective theory, that is the $U(N)$ principal chiral model.

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1. Introduction

Superconductivity is one of the most important phenomena in condensed matter physics. A Josephson junction is made of an insulator sandwiched by two superconductors with condensates Ψ_1 and Ψ_2 , in which the tunneling effect introduces a Josephson term $\Psi_1^* \Psi_2$. When a magnetic field is applied to a type-II superconductor the magnetic flux is squeezed into vortices. In the case of a Josephson junction of two type-II superconductors, vortices in the bulk are absorbed into the insulator, becoming Josephson vortices or fluxons; see Ref. [1] as a review. It is known

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that dynamics of Josephson vortices can be described by the sine-Gordon equation. Josephson vortices also appear in high- T_c superconductors with multi-layered structures [2] and in two coupled Bose–Einstein condensates [3]. It is also known that not only Josephson junctions but also multi-band superconductors have intrinsic Josephson terms to allow sine-Gordon solitons [4–7].

In a field theory framework, a flexible Josephson junction was proposed in Ref. [8], in which the Josephson junction is represented by a domain wall reducing to the usual Josephson junction in the heavy domain wall limit. Vortices in the bulk become sine-Gordon kinks inside the domain wall [8,9] as usual Josephson junctions. In the strong gauge coupling limit, the model is reduced to the \mathbb{CP}^1 model, and the domain wall is reduced to a \mathbb{CP}^1 domain wall [10,11] which is also magnetic domain wall in ferromagnets e.g. [12]. The Josephson term introduces the sine-Gordon potential in the effective theory of the $d = 1 + 1$ dimensional domain wall world-volume.

In this paper, we discuss a Josephson junction of non-Abelian color superconductors and non-Abelian Josephson vortices in it. Non-Abelian superconductors can be described by a $U(N)$ (or $SU(N)$) gauge theory with N scalar fields in the fundamental representation. $U(N)$ gauge symmetry is spontaneously broken, and therefore is referred to as a non-Abelian or color superconductor. One example is supersymmetric gauge theories in the Higgs phase, studied extensively in these years [13–15]. The other example is the color–flavor locked phase of QCD at extremely high density [16,17]. We discuss physics that appears when two such non-Abelian superconductors are connected as a non-Abelian Josephson junction that is flexible. Instead of the domain wall of the above $U(1)$ superconductors, a non-Abelian Josephson junction can be represented by a non-Abelian domain wall [18–20]. On the other hand, vortices in non-Abelian superconductors are non-Abelian vortices or color magnetic flux tubes. Non-Abelian vortices were first found in supersymmetric theories [21–24], and they carry non-Abelian \mathbb{CP}^{N-1} moduli; see Refs. [13–15] for a review. Non-Abelian vortices in high density QCD were found in Ref. [25] and they also carry \mathbb{CP}^2 moduli [26]; see Ref. [17] for a review. Therefore, when a non-Abelian vortex is placed parallel to a non-Abelian vortex, the former is absorbed into the latter to minimize the energy as parallel to usual Josephson junctions. In order to find what the fate of this non-Abelian vortex, we consider the effective theory approach. The effective field theory of a non-Abelian domain wall is the $U(N)$ chiral Lagrangian or principal chiral model [18–20]. We show that conventional (quadratic) non-Abelian Josephson term introduced in the model induces the conventional (modified or quadratic) pion mass term in the domain wall effective theory. This model has been recently found to admit a non-Abelian sine-Gordon soliton [27] that carries \mathbb{CP}^{N-1} moduli [28]. By calculating the non-Abelian color magnetic flux, we find that the non-Abelian sine-Gordon soliton is precisely the non-Abelian vortex that is absorbed into the non-Abelian Josephson junction (domain wall), so we call it non-Abelian Josephson vortex. For the quadratic Josephson term, we find that a non-Abelian sine-Gordon kink carries a half color magnetic flux of the non-Abelian vortex in the bulk. Therefore, one non-Abelian vortex is split into two color flux tubes inside the domain wall. We also discuss $3 + 1$ dimensional configurations of Josephson vortices, suggesting a non-Abelian extension of a D-brane soliton.

This paper is organized as follows. After our model is given in Section 2, we present the main results in Section 3; in the presence of a non-Abelian domain wall, we construct the domain wall effective theory by the moduli approximation to obtain the $U(N)$ chiral Lagrangian. We add the non-Abelian Josephson term in the original theory and find that it induces a pion mass term in the effective theory. We then construct a non-Abelian sine-Gordon soliton on the domain wall and show that it carries a non-Abelian magnetic flux, to show the coincidence between the non-Abelian sine-Gordon soliton and the non-Abelian vortex. Section 4 is devoted to a summary and discussion.

2. The model

We consider the $U(N)$ gauge theory coupled with two N by N charged complex scalar fields $H_1(x)$ and $H_2(x)$ summarized by $H = (H_1, H_2)$, with massless and real N by N scalar field $\Sigma(x)$ in $d = 2 + 1$ dimensions. The Lagrangian that we consider is given by

$$\mathcal{L} = -\frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \frac{1}{g_2^2} \text{tr} (D_\mu \Sigma)^2 + \text{tr} |D_\mu H|^2 - V \quad (1)$$

$$V = \frac{\lambda}{4} \text{tr} (HH^\dagger - v^2 \mathbf{1}_N)^2 + \text{tr} |\Sigma H - H M|^2 \quad (2)$$

where the covariant derivatives are $D_\mu H = \partial_\mu H - i A_\mu H$ and $D_\mu \Sigma = \partial_\mu \Sigma - i[A_\mu, \Sigma]$, g is the gauge coupling constant common for $U(1)$ and $SU(N)$ factors, g_2 and λ are coupling constants, and the masses of H are given by $M = \text{diag}(m_1 \mathbf{1}_N, m_2 \mathbf{1}_N)$ with $m_1 > m_2$. The symmetry of the model is $U(N)$ gauge (color) symmetry and global (flavor) symmetries

$$A_\mu \rightarrow g A_\mu g^{-1} + i g \partial_\mu g^{-1}, \quad \Sigma \rightarrow g \Sigma g^{-1}, \quad H \rightarrow g H, \quad g \in U(N)_C \quad (3)$$

$$H_1 \rightarrow H_1 U_L, \quad H_2 \rightarrow H_2 U_R, \quad U_{L,R} \in SU(N)_{L,R}. \quad (4)$$

In the limit $g_2 = g$, $\lambda/2 = g^2$, the model enjoys $\mathcal{N} = 4$ supersymmetry (with eight supercharges) in $d = 2 + 1$ by doubling scalar fields H and adding fermion superpartners; see, e.g., Ref. [14] for a review. In this paper, supersymmetry is not essential apart from technical reasons.

When $g_2 \rightarrow \infty$, the kinetic term of Σ disappears, then it becomes an auxiliary field that can be eliminated by its equation of motion as

$$\Sigma = \frac{H M H^\dagger}{H H^\dagger}. \quad (5)$$

If we further take $\lambda \rightarrow \infty$, the above Lagrangian is reduced to

$$\mathcal{L} = -\frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \text{tr} |D_\mu H|^2 - V \quad (6)$$

$$V = -\text{tr} [v^{-2} (H M H^\dagger)^2 + H M^2 H^\dagger] = \frac{4m^2}{v^2} \text{tr} (H_1 H_1^\dagger H_2 H_2^\dagger). \quad (7)$$

Without taking any limits, we may consider the Lagrangian

$$\mathcal{L}_2 = -\frac{1}{4g^2} \text{tr} F_{\mu\nu} F^{\mu\nu} + \text{tr} |D_\mu H|^2 - V_2 \quad (8)$$

$$V_2 = \frac{\lambda}{4} \text{tr} (H H^\dagger - v^2 \mathbf{1}_N)^2 + \frac{4m^2}{v^2} \text{tr} (H_1 H_1^\dagger H_2 H_2^\dagger) \quad (9)$$

from the beginning, instead of the above procedure.

Let us discuss the vacuum structure of the model. In the massless case $m_1 = m_2 = 0$, the flavor symmetry is enhanced to $SU(2N)$, and the vacuum can be taken to be

$$H = (v \mathbf{1}_N, \mathbf{0}_N), \quad \Sigma = \mathbf{0}_N \quad (10)$$

by using the $SU(2N)$ flavor symmetry. This is the so-called color–flavor locked vacuum. The moduli space of vacua is the Grassmann manifold, see, e.g., Ref. [29]:

$$Gr_{2N,N} \simeq \frac{SU(2N)}{SU(N) \times SU(N) \times U(1)}. \quad (11)$$

In the massive case, $m_1, m_2 \neq 0$, the vacuum is split into disjoint vacua

$$H = (v\mathbf{1}_N, \mathbf{0}_N), \quad \Sigma = m_1\mathbf{1}_N, \quad \text{or} \quad H = (\mathbf{0}_N, v\mathbf{1}_N), \quad \Sigma = m_2\mathbf{1}_N, \quad (12)$$

with the following unbroken *global* symmetries, respectively¹:

$$SU(N)_{C+L}, \quad \text{or} \quad SU(N)_{C+R}. \quad (13)$$

Each vacuum given here can be interpreted as a non-Abelian superconductor, since gauge group $U(N)$ is spontaneously broken.

For explicit calculation, we work in the strong gauge coupling limit $g^2 \rightarrow \infty$ in which the gauge field becomes non-dynamical and can be eliminated by its equation of motions as

$$A_\mu = \frac{i}{2}v^{-2}[H\partial_\mu H^\dagger - (\partial_\mu H)H^\dagger]. \quad (14)$$

The model reduces to a Grassmann sigma model with the target space in Eq. (11) and a potential term, known as the massive Grassmann model [30]. We denote the limit $\lambda/2 = g^2 = g_2^2 \rightarrow \infty$ the sigma model limit. Although we take this limit for explicit calculation, the results in this paper do not rely on this limit.

As we will see in the next subsection, the above model admits a domain wall solution interpolating the two vacua in Eq. (12), that separate the condensation H_1 and H_2 . In order to interpret this domain wall as a Josephson junction, we now introduce a deformation

$$\mathcal{L}_{J,1} = -\gamma \text{tr}(H_1^\dagger H_2 + H_2^\dagger H_1) \quad (15)$$

that explicitly breaks the flavor symmetry in Eq. (4) to $SU(N)_{L+R}$. We refer this term the non-Abelian “Josephson” interaction term [8,27], because it is a non-Abelian matrix extension of a Josephson term in the Josephson junction of two superconductors with two condensations. In the case of supersymmetric extension of the model, the Josephson term in Eq. (15) breaks supersymmetry explicitly but supersymmetry is not essential in our study. Instead of the Josephson term in Eq. (15), we may also consider a quadratic Josephson-like term

$$\mathcal{L}_{J,2} = -\gamma \text{tr}[(H_1^\dagger H_2)^2 + (H_2^\dagger H_1)^2]. \quad (16)$$

This is a non-Abelian extension of the Josephson-like term in chiral p-wave superconductors [31,32]. We refer it the non-Abelian quadratic Josephson term.

3. Non-Abelian Josephson vortex: non-Abelian sine-Gordon soliton inside a non-Abelian domain wall

3.1. Non-Abelian domain wall

Meanwhile we consider the case in the absence of the Josephson term $\gamma = 0$, and we turn it on later. A non-Abelian domain wall solution interpolating between the left and right vacua, which is placed perpendicular to the x^2 coordinate, is given by [18–20,33]

¹ Exactly speaking, there are $N - 2$ more vacua between these two vacua in the supersymmetric theory. Those are not relevant in our context.

$$H_{\text{wall},0} = \frac{v}{\sqrt{1 + |u_{\text{wall}}|^2}} (\mathbf{1}_N, u_{\text{wall}} \mathbf{1}_N), \quad u_{\text{wall}}(x^2) = e^{\mp m(x^2 - Y) + i\varphi},$$

$$\Sigma_{\text{wall},0} = v^{-2} H M H^\dagger, \quad A_{2,\text{wall},0} = \frac{i}{2} v^{-2} [H \partial_2 H^\dagger - (\partial_2 H) H^\dagger], \quad (17)$$

in the sigma model limit. The most general solution can be obtained by acting the $SU(N)_{\text{C+L+R}}$ symmetry on the particular solution in Eq. (17):

$$H_{\text{wall}} = V H_{\text{wall},0} \begin{pmatrix} V^\dagger & 0 \\ 0 & V \end{pmatrix} = \frac{v}{\sqrt{1 + e^{\mp 2m(x^2 - Y)}}} (\mathbf{1}_N, e^{\mp m(x^2 - Y)} U),$$

$$\Sigma_{\text{wall}} = V \Sigma_{\text{wall},0} V^\dagger, \quad A_{2,\text{wall}} = V A_{2,\text{wall},0} V^\dagger, \quad (18)$$

with $V \in SU(N)$ and $U \equiv V^2 e^{i\varphi} \in U(N)$. Therefore, the domain wall has the moduli²

$$\mathcal{M}_{\text{wall}} \simeq \mathbb{R} \times U(N). \quad (19)$$

In the presence of the Josephson term $\gamma \neq 0$, this domain wall behaves as a Josephson junction as we will see below.

3.2. Non-Abelian vortex

Let us discuss a non-Abelian vortex. In the left vacuum in Eq. (12), we can neglect H_2 . There, $U(N)$ symmetry is spontaneously broken and is locked with the $SU(N)_{\text{L}}$ flavor symmetry to be the $SU(N)_{\text{C+L}}$ color–flavor locked symmetry. There is a non-Abelian vortex solution with a non-Abelian magnetic field and a scalar field given by

$$F_0 = \text{diag}(F_*(r), 0, \dots, 0), \quad H_0 = v \text{diag}(f(r)e^{i\theta}, 1, \dots, 1), \quad (20)$$

respectively, with the boundary conditions $f(r) \rightarrow 1$ ($r \rightarrow \infty$) and $f(r) \rightarrow 0$ ($r = 0$), where (r, θ) are cylindrical coordinates. This solution is typically obtained by embedding of the Abrikosov–Nielsen–Olesen (ANO) vortex solution [34] ($F_*(r), f(r)e^{i\theta}$) in the upper-left corner. The integral of the magnetic field is quantized:

$$\int d^2x F_0 = \text{diag}(2\pi, 0, \dots, 0). \quad (21)$$

The most general solution is obtained by acting the color–flavor locked symmetry $SU(N)_{\text{C+L}}$:

$$F = V \text{diag}(F_*(r), 0, \dots, 0) V^\dagger, \quad H = v V \text{diag}(f(r)e^{i\theta}, 1, \dots, 1) V^\dagger,$$

$$V \in SU(N). \quad (22)$$

In other words, the solution spontaneously breaks the color–flavor locked symmetry $SU(N)_{\text{C+L}}$ into a subgroup $SU(N-1) \times U(1)$, and consequently there appears moduli localized on the vortex core;

$$\mathcal{M}_{\text{vortex}} \simeq \mathbb{C} \times \mathbb{C}P^{N-1} = \mathbb{C} \times \frac{SU(N)_{\text{C+L}}}{SU(N-1) \times U(1)}. \quad (23)$$

We can repeat the same for the right vacuum. The Josephson term does not affect the non-Abelian vortex living in each vacuum. When scalar fields H are massless, vortices are so-called

² In the supersymmetric extension, the full moduli space is the cotangent bundle over $U(N)$, $T^*U(N)$ larger than that in Eq. (19).

non-Abelian semi-local vortices [35], which reduce to Grassmann lumps in the sigma model limit. In the presence of the mass term as we are discussing here, vortices are local vortices.

If we consider a non-Abelian vortex parallel to a non-Abelian vortex, the latter will be absorbed into the former to minimize the total energy, in analogy with usual superconductors. In this case, the Josephson term plays an essential role. A question is what the fate of the vortex if it is absorbed into the wall.

One comment is in order here. In the following, we will take the sigma model limit for concrete calculations. In this limit, the bulk Grassmann lumps become singular (a delta function) in the presence of the mass term, known as small lump singularity. Nevertheless, they can live stably inside the domain wall, as we will show below. The singularity appearing in the bulk lumps is an artifact of the sigma model limit, and small lumps are replaced by the ANO vortices without taking that limit.

3.3. Low-energy effective theory on non-Abelian domain wall world-volume

By using the moduli approximation [36,37], the effective theory of the domain wall can be constructed by promoting the moduli X and U to fields $X(x^i)$ and $U(x^i)$, respectively ($i = 0, 1$) on the world volume of the domain wall. The result is [18–20]:

$$\mathcal{L}_{\text{wall}} = -\frac{v^2}{4m} \text{tr} \left(U^\dagger \partial_i U U^\dagger \partial^i U \right) + \frac{v^2}{2m} \partial_i X \partial^i X. \quad (24)$$

Apart from the position modulus X , this is the $U(N)$ chiral Lagrangian or principal chiral model.

Here we turn on the Josephson term perturbatively in the regime $\gamma \ll m$, and consider the effect on the domain wall effective action. We thus obtain

$$\begin{aligned} \mathcal{L}_{\text{wall}, J, 1} &= -\gamma \int_{-\infty}^{+\infty} dx^2 \frac{e^{\mp m(x^2 - Y)}}{1 + e^{\mp 2m(x^2 - Y)}} (\text{tr} U + \text{tr} U^\dagger) = -\frac{\pi \gamma}{2m} (\text{tr} U + \text{tr} U^\dagger) \\ &= -m'^2 (\text{tr} U + \text{tr} U^\dagger), \quad m'^2 \equiv \frac{\pi \gamma}{2m} \end{aligned} \quad (25)$$

This term introduces the conventional pion mass term in the $U(N)$ chiral Lagrangian in Eq. (24). The $U(N)$ target space is lifted by this potential term, leaving the unique vacuum

$$U = \mathbf{1}_N. \quad (26)$$

In the presence of the non-Abelian quadratic Josephson term in Eq. (16) instead of the linear term, the following term is induced in the domain wall effective theory:

$$\begin{aligned} \mathcal{L}_{\text{wall}, J, 2} &= -\gamma_2 \int_{-\infty}^{+\infty} dx^2 \left(\frac{e^{\mp m(x^2 - Y)}}{1 + e^{\mp 2m(x^2 - Y)}} \right)^2 (\text{tr} U^2 + \text{tr} U^{\dagger 2}) = -\frac{\pi \gamma_2}{4m} (\text{tr} U^2 + \text{tr} U^{\dagger 2}) \\ &= -m_2'^2 (\text{tr} U^2 + \text{tr} U^{\dagger 2}), \quad m_2'^2 \equiv \frac{\pi \gamma_2}{4m}. \end{aligned} \quad (27)$$

This mass term is sometimes called a modified pion mass in the context of the $SU(2)$ Skyrme model [38,39]. In this case, there are two discrete vacua

$$U = \pm \mathbf{1}_N. \quad (28)$$

There exists a domain wall interpolating them [39].

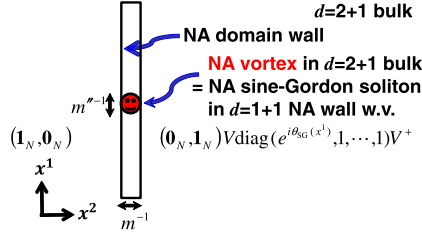


Fig. 1. A schematic picture of a non-Abelian sine-Gordon soliton in a non-Abelian domain wall describing a non-Abelian vortex.

3.4. Non-Abelian sine-Gordon soliton inside a non-Abelian domain wall

The $U(N)$ chiral Lagrangian in Eq. (24) with the mass term in Eq. (25) admits non-Abelian sine-Gordon solitons [27,28]. A single soliton solution is

$$U(x) = \text{diag}(u(x^1), 1, \dots, 1), \quad (29)$$

$$u(x^1) = \exp i\theta_{\text{SG}}(x^1) = \exp\left(4i \arctan \exp[m''(x^1 - X)]\right), \quad m''^2 \equiv 2\pi\gamma/v^2. \quad (30)$$

The most general solution is given by acting the $SU(N)$ symmetry on it:

$$U(x) = V \text{diag}(u(x^1), 1, \dots, 1) V^\dagger, \quad V \in SU(N). \quad (31)$$

By removing the redundancy, V takes a value in a coset space

$$V \in \frac{SU(N)}{SU(N-1) \times U(1)} \simeq \mathbb{C}P^{N-1}, \quad (32)$$

and consequently the moduli space of the sine-Gordon soliton is found to be

$$\mathcal{M}_{\text{SG soliton}} = \mathbb{R} \times \mathbb{C}P^{N-1}. \quad (33)$$

The total composite configuration is therefore:

$$H_{\text{composite}} = \frac{1}{\sqrt{1 + e^{\mp 2m(x^2 - Y)}}} \left(\mathbf{1}_N, e^{\mp m(x^2 - Y)} V \text{diag}(e^{i\theta_{\text{SG}}(x^1)}, 1, \dots, 1) V^\dagger \right). \quad (34)$$

This configuration with the upper sign goes to

$$H_{\text{composite}} \rightarrow \begin{cases} (\mathbf{1}_N, \mathbf{0}_N), & x^2 \rightarrow -\infty \\ (\mathbf{0}_N, \mathbf{1}_N) V (e^{i\theta_{\text{SG}}(x^1)}, 1, \dots, 1) V^\dagger, & x^2 \rightarrow +\infty \end{cases} \quad (35)$$

What is this solution in the $d = 2 + 1$ dimensional bulk theory? Our claim is that it is precisely a non-Abelian vortex, as illustrated in Fig. 1. The agreement between them is not only the moduli in Eq. (23) for a vortex and Eq. (33) for a sine-Gordon soliton but also the non-Abelian fluxes ($a = 1, 2$):

$$\begin{aligned} \int d^2x F_{12} &= \oint dx^a A_a = \oint dx^a \frac{i}{2} v^{-2} [H \partial_\mu H^\dagger - (\partial_\mu H) H^\dagger] \\ &= V \text{diag} \left(\int_{-\infty}^{+\infty} dx^1 \partial_1 \theta_{\text{SG}}|_{x^2=+\infty}, 0, \dots, 0 \right) V^\dagger \end{aligned}$$

$$\begin{aligned}
&= V \text{diag} \left([\theta_{\text{SG}}]_{(x^1, x^2)=(-\infty, +\infty)}^{(x^1, x^2)=(+\infty, +\infty)}, 0, \dots, 0 \right) V^\dagger \\
&= V \text{diag}(2\pi k, 0, \dots, 0) V^\dagger.
\end{aligned} \tag{36}$$

Here we have used the sigma model limit in the second equality and Eq. (35) for the third equality, and have assumed the k winding of the phase θ_{SG} for k sine-Gordon solitons in the last equality. The flux in Eq. (36) coincides with that of the non-Abelian vortex in Eq. (20) showing the one-to-one correspondence between the $\mathbb{C}P^{N-1}$ moduli of the sine-Gordon soliton and the non-Abelian vortex. The flux matching in Eq. (36) also shows the coincidence of the topological charges of them:

$$T_{\text{vortex}} = \int d^2x \text{tr} F_{12} = 2\pi k. \tag{37}$$

These precisely proves the identification of a non-Abelian vortex and a non-Abelian sine-Gordon kink inside the non-Abelian domain wall.

The $U(N)$ chiral Lagrangian in Eq. (24) with the quadratic pion mass term in Eq. (27) also admits non-Abelian sine-Gordon solitons [27]. In this case, the modified sine-Gordon soliton

$$u(x) = e^{i\theta_{\text{SG},2}(x^1)} = \exp(2i \arctan \exp \sqrt{2} m_2''(x^1 - X)), \quad m_2''^2 \equiv \frac{\pi \gamma_2}{v^2} \tag{38}$$

is embedded into U in Eq. (29). Note that the range of θ for a single soliton is half the conventional sine-Gordon soliton. In the total configuration the conventional sine-Gordon soliton θ_{SG} in Eq. (34) is replaced by $\theta_{\text{SG},2}$ given here. From the same calculation in Eq. (36), we obtain

$$\begin{aligned}
\int d^2x F_{12} &= V \text{diag} \left([\theta_{\text{SG},2}]_{(x^1, x^2)=(-\infty, +\infty)}^{(x^1, x^2)=(+\infty, +\infty)}, 0, \dots, 0 \right) V^\dagger \\
&= V \text{diag}(\pi, 0, \dots, 0) V^\dagger,
\end{aligned} \tag{39}$$

for the single soliton. We thus have found that this solution carries a half color flux of a single non-Abelian vortex in the bulk. Therefore, one non-Abelian vortex must be split into two fractional non-Abelian fluxes when absorbed into the domain wall. This is a non-Abelian extension of Ref. [40].

In the absence of any Josephson term, non-Abelian fluxes are diluted to infinity when absorbed into the domain wall, since the size of non-Abelian sine-Gordon soliton m'^{-1} or $m_2'^{-1}$ goes to infinity in the limit of γ to zero.

3.5. $3+1$ dimensional configurations

We have been discussing configurations in dimension $d = 2 + 1$. In $d = 3 + 1$, a non-Abelian vortex can end on a non-Abelian domain wall when they are perpendicular to each other [18], which is a non-Abelian generalization of a D-brane soliton [41]. In the presence of the conventional non-Abelian Josephson term in Eq. (15), a non-Abelian flux ending on a non-Abelian domain wall turns to a non-Abelian sine-Gordon soliton inside the wall, and it can escape to the other side of the domain wall, as illustrated in Fig. 2.

On the other hand, in the presence of the quadratic non-Abelian Josephson term in Eq. (16), a non-Abelian flux ending on a non-Abelian domain wall is split into two fractional non-Abelian sine-Gordon solitons that separate the two vacua $U = \pm 1$, as illustrated in Fig. 3(a). They have to join when they escape to the other side of the domain wall, as illustrated in Fig. 3(b). Consequently, there appears a domain wall ring that separates two vacua in $U = \pm \mathbf{1}_N$ with two vortices on it. This structure in $d = 2 + 1$ is in fact known for the Abelian case in chiral p-wave superconductors [32].

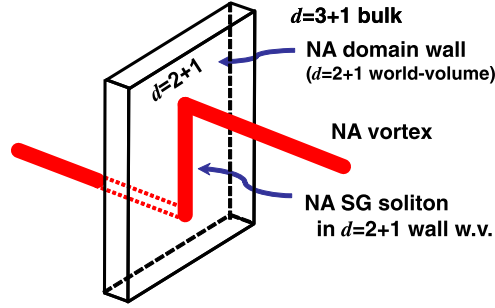


Fig. 2. Three dimensional configurations of vortices on Josephson junctions with a linear Josephson term. A non-Abelian vortex ends on a non-Abelian domain wall, becomes a non-Abelian sine-Gordon soliton inside the domain wall, and escapes to the other side of domain wall.

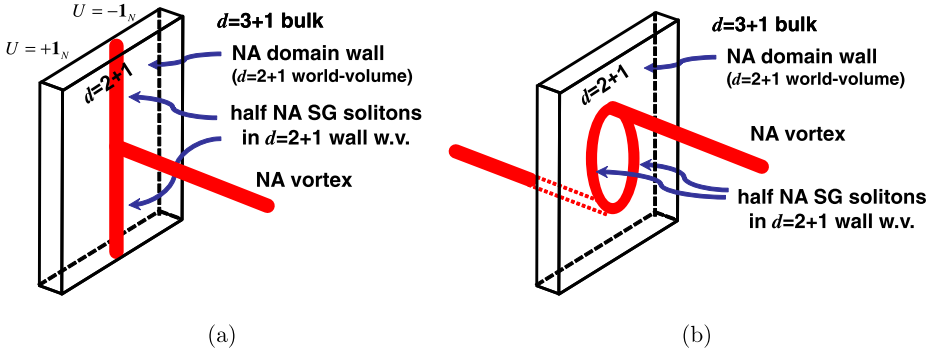


Fig. 3. Three dimensional configurations of vortices on Josephson junctions with a quadratic Josephson term. (a) A non-Abelian vortex ending on a non-Abelian domain wall splits into two fractional fluxes inside the domain wall, represented as half non-Abelian sine-Gordon solitons. (b) They join to escape to the other side of domain wall, leaving a domain wall ring.

In either case, when vortices do not end on both sides of the domain wall, the domain wall is logarithmically bent. When two vortices end on the domain wall from the both sides there is a linear confinement force coming from the sine-Gordon soliton(s) between the two endpoints. Consequently, the vortices on the both sides tend to join at the same endpoint, if they are orthogonal to the domain wall. If the vortices and the domain wall have an angle, the two endpoints will be separated and a vortex kink is formed, as is known for the Abelian Josephson junctions [2].

4. Summary and discussion

In this paper, we have constructed the non-Abelian Josephson vortex in a (flexible) non-Abelian Josephson junction of two non-Abelian color superconductors. The effective field theory of a non-Abelian domain wall is the $U(N)$ chiral Lagrangian that has the conventional (modified or quadratic) pion mass term when the linear (quadratic) non-Abelian Josephson term is introduced in the model. Then, we have shown that a non-Abelian sine-Gordon soliton found in Ref. [27] carries a (half) non-Abelian magnetic flux of a non-Abelian vortex in the bulk for the linear (quadratic) Josephson term, and the $\mathbb{C}P^{N-1}$ moduli. We thus have found that a non-Abelian vortex becomes one (two) non-Abelian sine-Gordon soliton(s) when absorbed into the

non-Abelian Josephson junction. We have also discussed $3 + 1$ dimensional configurations of Josephson vortices, that constitute a non-Abelian extension of a D-brane soliton.

Monopoles and Yang–Mills instantons trapped in the non-Abelian Josephson junction have been studied in Ref. [42]. They are expressed as global vortices and Skyrmions, respectively, in the domain wall effective theory.

The $U(N)$ chiral Lagrangian appearing as the low-energy theory of QCD receives an axial anomaly for the $U(1)$ part and has a potential term along the $U(1)$ direction. In this case, there appear multiple domain walls other than the non-Abelian sine-Gordon soliton [43]. On the other hand, the $U(N)$ chiral Lagrangian arising in the non-Abelian domain wall discussed in this paper does not have such a term.

In this paper, we have considered $U(N)$ gauge theory but $SU(N)$ gauge group does not change the main results. We still have a flux matching between a non-Abelian sine-Gordon soliton inside the non-Abelian domain wall and a non-Abelian vortex in the bulk. Therefore, the results in this paper can be applied to color superconductors of high density quark matter. If quark matter is separated by an insulator for instance by some modulation such as crystalline superconductivity, it will give non-Abelian Josephson junctions. Non-Abelian vortices there become non-Abelian Josephson vortices by trapped to insulating regions.

It was already pointed out in Ref. [27] that the principal chiral model with arbitrary groups G in the form of $\frac{G \times U(1)}{\mathbb{Z}_r}$ admits non-Abelian G sine-Gordon solitons. Non-Abelian vortices with this type of gauge groups were also found before [44], such as $SO(N)$ and $USp(2N)$ groups [45]. Therefore, by changing $U(N)$ groups in our model to $\frac{G \times U(1)}{\mathbb{Z}_r}$, there should exist a non-Abelian domain wall whose effective theory is the principal chiral model with $\frac{G \times U(1)}{\mathbb{Z}_r}$, and non-Abelian vortices will become non-Abelian G Josephson vortices as non-Abelian G sine-Gordon solitons in the domain wall effective theory, if they are absorbed into the domain wall.

One may construct a Josephson junction of three superconductors that meet at one point, where the insulator is of a Y shape. As for a flexible version of this three junction, we can use a domain wall junction solution for which exact solution is available [46].

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